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REPRESENTATION OF L_p-NORMS AND ISOMETRIC EMBEDDING IN L_p-SPACES

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ABSTRACT

For fixed $1 \leqq p < \infty$ the L_p-semi-norms on $\mathbf{R}^n$ are identified with positive linear functionals on the closed linear subspace of $C(\mathbf{R}^n)$ spanned by the functions $|\langle \xi, \cdot \rangle|^p$, $\xi \in \mathbf{R}^n$. For every positive linear functional σ, on that space, the function $\phi_\sigma : \mathbf{R}^n \to \mathbf{R}$ given by $\phi_\sigma(\xi) = \sigma(|\langle \xi, \cdot \rangle|^p)^{1/p}$ is an L_p-semi-norm and the mapping $\sigma \to \phi_\sigma$ is 1-1 and onto. The closed linear span of $|\langle \xi, \cdot \rangle|^p$, $\xi \in \mathbf{R}^n$ is the space of all even continuous functions that are homogeneous of degree p, if p is not an even integer and is the space of all homogeneous polynomials of degree p when p is an even integer. This representation is used to prove that there is no finite list of norm inequalities that characterizes linear isometric embeddability, in any L_p unless $p = 2$.

1. Introduction

A function $\phi : \mathbf{R}^n \to \mathbf{R}$ is called an *L_p-semi-norm* if there is a linear T from $\mathbf{R}^n$ to a Banach space $L_p(\mu)$ for some measure μ such that $\| T\xi \| = \phi(\xi)$ for all ξ in $\mathbf{R}^n$. (Equivalently if there is a 1-1 linear T from $\mathbf{R}^n$ to the semi-normed space $\mathscr{L}_p(0,1)$ with $\| T\xi \|_p = \phi(\xi)$.) It is called an *L_p-norm* if T is 1-1. The L_p-semi-norms on $\mathbf{R}^n$ are identified (Theorem 2.3) with the positive linear functionals on the space H_p^n — the closed linear subspace of $C(\mathbf{R}^n)$ spanned by the functions $|\langle \xi, \cdot \rangle|^p$, $\xi \in \mathbf{R}^n$, where $\langle \cdot , \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$. For every positive linear functional σ on H_p^n the function $\phi_\sigma : \mathbf{R}^n \to \mathbf{R}$ given by $\phi_\sigma(\xi) = (\sigma(|\langle \xi, \cdot \rangle|^p))^{1/p}$ is an L_p-semi-norm and the mapping $\sigma \to \phi_\sigma$ is 1-1 and onto all L_p-semi-norms. (The mapping $\sigma \to \phi_\sigma^p$ is linear.)

It is proved that if $0 < p < \infty$ with p not an even integer, then H_p^n is the space of all even continuous functions that are homogeneous of degree p (Theorem 2.1), and that if p is an even integer then H_p^n is the space of all homogeneous

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polynomials of degree p (Theorem 2.2). This provides a representation of L_p-semi-norms on $\mathbf{R}^n$ by even positive Borel measures on S^{n-1}. The identification of the L_p-norms on $\mathbf{R}^n$ is provided in Section 3 (Lemma 3.1) where we derive various results on embeddability into L_p-spaces: in particular we solve a problem of Witsenhousen concerning the characterization of embeddability into L_1-spaces. In fact it was Witsenhousen's problem, on whether or not every normed space, whose norm satisfies the quadrilateral property, is isometrically embeddable in L_1, that inspired this research.

A normed space (a space for short) $(X, \|\ \|)$ is *isometrically embeddable* (*embeddable* for short) in the normed space $(Y, \|\ \|)$ if there is a linear, one-to-one, $T: X \to Y$ with $\|Tx\| = \|x\|$. It is *embeddable* in L_p (some fixed $1 \leqq p < \infty$) if there is a measure space (Ω, Σ, μ) such that $(X, \|\ \|)$ is embeddable in $L_p(\Omega, \Sigma, \mu)$ and we say that $\|\ \|$ is an L_p-norm. A normed space $(X, \|\ \|)$ is embeddable in L_p iff all of its finite-dimensional subspaces are embeddable in L_p. Thus we restrict our attention to embeddability of finite-dimensional spaces.

It is well known [3, 4] that a space $(X, \|\ \|)$ is embeddable in L_2 if, and only if, for all x and y in X

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1.1}$$

For other values of p, i.e., $p \neq 2$, it was not known whether there is a single "norm inequality" (or "norm equality") that characterizes embeddability in L_p. Levy [6] proved that a space $(X, \|\ \|)$ is embeddable in L_p, $1 \leqq p \leqq 2$, if, and only if, for all n and for all $x_1, \dots, x_n$ in X,

$$\sum_{i,j} \|x_i - x_j\|^p \alpha_i \alpha_j \leqq 0 \tag{1.2}$$

for all scalars α_i, $1 \leqq i \leqq n$, with $\Sigma\, \alpha_i = 0$. Therefore we have an infinite list of "norm inequalities" that characterizes embeddability in L_p, $1 \leqq p < 2$.

Krivine [5] has found an infinite list of "norm inequalities" which characterizes embeddability in L_p for $p > 2$ with p different from an even integer; a normed space $(X, \|\ \|)$ is embeddable in L_p with $p \geqq 1$, $2r - 2 < p < 2r \leqq 4k$ for some positive integers r and k, if, and only if, for $x_1, \dots, x_n$ in X and every choice of scalars $\alpha_1, \dots, \alpha_n$ with $\Sigma_{i=1}^n \alpha_i = 0$, we have that

$$(-1)^r \sum_{\pm, i_1, \dots, i_{2k}} \|x_{i_1} \pm x_{i_2} \pm x_{i_3} \pm \cdots \pm x_{i_{2k}}\|^p \alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \cdots \alpha_{i_{2k}} \geqq 0 \tag{1.3}$$

where in the sum Σ, $i_1, i_2, \dots, i_{2k}$ vary independently from 1 to n and the sum is extendent also over all possible choices of signs (the number of terms in Σ is, thus, $n^{2k} \cdot 2^{2k-1}$).

Embeddability in L_1 is closely related to the theory of zonoids — ranges of nonatomic vector measures. A finite-dimensional space $(\mathbf{R}^n, \| \ \|) \equiv (\mathbf{R}^n, \phi)$ is embeddable in L_1 if and only if ϕ is the support function of a zonoid whose center of symmetry is 0. Any sum of line segments $\Sigma_{i=1}^m [0, x_i]$ with $x_i \in \mathbf{R}^n$ is a zonoid. Such zonoids are called zonotopes and it is known that a compact subset of $\mathbf{R}^n$ is a zonoid if and only if it is a limit of zonotopes. Alexandrov [1] characterized the zonotopes as those polytopes whose faces are centrally symmetric. This is equivalent to the following result concerning embeddability in L_1: if $\| \ \|$ is a piecewise linear norm on $\mathbf{R}^n$, i.e., $B = \{x \in \mathbf{R}^n : \|x\| \leqq 1\}$ is a polytope, then $(\mathbf{R}^n, \| \ \|)$ is embeddable in L_1 if and only if all the faces of the dual set B^* are centrally symmetric. Witsenhausen [13] proved that a convex polytope (containing 0) is a zonotope if, and only if, its support function satisfies Hlawka's inequality, or equivalently a finite-dimensional real space with piecewise linear norm is embeddable in L_1, if and only if it has the quadrilateral property

$$\|x\| + \|y\| + \|z\| + \|x + y + z\| \geqq \|x + y\| + \|x + z\| + \|y + z\|. \tag{1.4}$$

Since every L_1 norm is a limit of piecewise linear L_1-norms, Witsenhausen raised the natural question as to whether or not the quadrilateral property characterizes embeddability in L_1. One of the results of this paper answers this question in the negative. In fact we prove that there is no finite list of inequalities that characterizes embeddability, neither in L_1 nor in any other L_p unless obviously $p = 2$. This is accomplished by proving that for every $1 \leqq p < \infty$, $p \neq 2$, and every $k \geqq 2$ there is a $(k+1)$-dimensional normed space that is not embeddable in L_p and all of its k-dimensional subspaces are embeddable in L_p. The result for L_1 was observed by the author at the annual meeting of the A.M.S. in January 1981 at San Francisco, and independently by W. Weil [12].

2. Representation of L_p-semi-norms

Let H_p^n be the closed linear space of functions in $C(\mathbf{R}^n)$ spanned by the functions $|\langle \xi, \cdot \rangle|^p$, $\xi \in \mathbf{R}^n$ where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$. As each function $|\langle \xi, \cdot \rangle|^p$ is even and homogeneous of degree p, so are all functions in H_p^n. Therefore there is a natural embedding, $T_p : H_p^n \to C_E(S^{n-1}) \equiv C(P^{n-1})$, of H_p^n into the space of all continuous even functions on the unit sphere, or equivalently into the space of continuous functions on the $(n-1)$-dimensional projective space, defined by $(T_p f)(x) = f(x)$.

THEOREM 2.1. *If $0 < p < \infty$ is not an even integer then H_p^n is the space of all* *even continuous functions that are homogeneous of degree* p; *equivalently* $T_p(H_p^n) = C_E(S^{n-1})$.

PROOF. Let μ be the orthogonal invariant measure on S^{n-1}. Then for every linear transformation $A:\mathbf{R}^n \to \mathbf{R}^n$ the function

$$\begin{aligned} f_A(x) &= \int_{S^{n-1}} |\langle A^*\xi, x\rangle|^p d\mu(\xi) \\ &= \int_{S^{n-1}} |\langle \xi, Ax\rangle|^p d\mu(\xi) \\ &= K\|Ax\|_2^p \\ &= K\langle Ax, Ax\rangle^{p/2} \\ &= K\langle x, A^*Ax\rangle^{p/2} \\ &= K\Big(\sum_{i,j} b_{ij}x_ix_j\Big)^{p/2} \end{aligned}$$

where (b_{ij}) is the matrix A^*A, is in H_p^n. As all symmetric matrices (b_{ij}) that are in a sufficiently small neighborhood of the identity are of the form A^*A for some matrix A it follows that for all symmetric matrices (b_{ij}) that are sufficiently close to the identity matrix, $(\Sigma_{i,j} b_{ij}x_ix_j)^{p/2}$ is in H_p^n. By differentiating with respect to b_{ij} it follows by induction that H_p^n contains all functions of the form

$$(*) \qquad Q(x_1,\dots,x_n)\Big(\sum_{i,j} b_{ij}x_ix_j\Big)^{p/2-k}$$

where Q is an homogeneous polynomial of degree $2k$ and (b_{ij}) is a symmetric matrix sufficiently close to the identity matrix. (Observe that we use here the fact that p is not an even integer and thus $p/2-k$ is never 0. If p is an even integer the conclusion is true for $k \leqq p/2$.) In particular, H_p^n contains the functions of the form $Q(x_1,\dots,x_n)\|x\|_2^{p-2k}$, where Q is an homogenous polynomial of degree $2k$.

Thus $T_p(H_p^n)$ contains all even polynomials and by the Stone–Weirstrass theorem it follows that $T_p(H_p^n) = C_E(S^{n-1})$. □

REMARKS. (i) For $p = 1$ the theorem has been proved by Rickert [9, 10] using spherical harmonics.

(ii) A slight variation of the preceding proof shows that there are no nontrivial closed subspaces of even continuous functions on $\mathbf{R}^n$ that are homogeneous of degree p, with p not an even integer, that are invariant under the general linear group $\mathrm{GL}(\mathbf{R}^n)$ (acting by $Af(x) = f(Ax)$).

THEOREM 2.2. *If p is an even integer then H_p^n is the space of all homogeneous polynomials of degree p.*

PROOF. Let $p = 2k$ be an even integer. Then each generating function $|\langle \xi, \cdot \rangle|^p = \langle \xi, \cdot \rangle^{2k}$ is an homogeneous polynomial of degree $p = 2k$. As the space of all homogeneous polynomials of degree p is closed it follows that H_p^n is contained in it. That each homogeneous polynomial of degree $p = 2k$ is a linear combination of functions of the form $\langle \xi, x \rangle^{2k}$ follows either by the argument of the preceding theorem (which implies that $T(H_{2k}^n)$ contains all even polynomials of degree $\leqq 2k$) or by the algebraic identity

$$l!\, x_1 x_2 \cdots x_l = \sum_{S \in 2^l} (-1)^{|S|-l} (x(S))^l$$

where 2^l is the power set of $\{1, \ldots, l\}$ and for $S \subset \{1, \ldots, l\}$, $x(S) = \Sigma_{i \in S} x_i$.

THEOREM 2.3. (i) *Every positive linear functional σ on H_p^n induces an $\mathcal{L}_p$-semi-norm ϕ_σ on $\mathbf{R}^n$ given by*

$$\phi_\sigma(\xi) = (\sigma(|\langle \xi, x \rangle|^p))^{1/p}.$$

(ii) *The mapping $\sigma \to \phi_\sigma$ from the set of positive linear functionals on H_p^n to the L_p-semi-norms on $\mathbf{R}^n$ is* 1-1 *and onto.*

PROOF. Assume that σ is a positive linear functional on H_p^n. Then by Theorems 2.1 and 2.2, the remark preceding them and the Riesz representation theorem there is a Borel measure on P^{n-1} or equivalently an even measure μ on S^{n-1} (if p is an even integer one extends first σ to a positive linear functional on $C_E(S^{n-1})$) such that

$$\sigma(|\langle \xi, \cdot \rangle|^p) = \int_{S^{n-1}} |\langle \xi, x \rangle|^p d\mu(x). \tag{2.3}$$

Let $\mathcal{L}_p = \mathcal{L}_p(S^{n-1}, \mathcal{B}, \mu)$ where $\mathcal{B}$ are Borel subsets of S^{n-1} and define a linear mapping $T: \mathbf{R}^n \to \mathcal{L}_p$ by $(Te_i)(x) = x_i$ where e_i are the unit vectors in $\mathbf{R}^n$ and $x = (x_1, \ldots, x_n) \in S^{n-1}$. Let $\xi = (\xi_1, \ldots, \xi_n) = \Sigma\, \xi_i e_i$. Then

$$\begin{aligned} \| T\xi \|_p &= \left(\int_{S^{n-1}} |(T\xi)(x)|^p d\mu(x) \right)^{1/p} \\ &= \left(\int_{S^{n-1}} |\langle \xi, x \rangle|^p d\mu(x) \right)^{1/p} \\ &= (\sigma(|\langle \xi, \cdot \rangle|^p))^{1/p} \\ &= \phi_\sigma(\xi). \end{aligned}$$

Thus ϕ_σ is an $\mathcal{L}_p$-semi-norm. To show that the mapping is onto, let ϕ be an $\mathcal{L}_p$-semi-norm on $\mathbf{R}^n$ and let T be an embedding of $(\mathbf{R}^n, \phi)$ into $\mathcal{L}_p(\Omega, \Sigma, \mu)$, $e_1, \cdots, e_n$ the basis of the unit vectors in $\mathbf{R}^n$. Define a measure η on $\mathbf{R}^n$ by $\eta(B) = \mu\{\omega \in \Omega : (Te_1(\omega), \ldots, Te_n(\omega)) \in B\}$. Then it is easily verified that $\int_{\mathbf{R}^n} \| x \|^p d\eta(x) < \infty$ and thus η defines a positive linear functional σ on H_p^n, given by $\sigma(f) = \int_{\mathbf{R}^n} f(y) d\eta(y)$. We have to show that $\phi_\sigma = \phi$. Let $\xi \in \mathbf{R}^n$. Then

$$\begin{aligned} \phi_\sigma^p(\xi) &= \sigma(|\langle \xi, \cdot \rangle|^p) \\ &= \int_{\mathbf{R}^n} |\langle \xi, x \rangle|^p d\eta(x) \\ &= \int_\Omega \left| \sum \xi_i Te_i(\omega) \right|^p d\mu(\omega) \\ &= \| T\xi \|_p^p \\ &= \phi^p(\xi). \end{aligned}$$ □

REMARK. An immediate corollary of Theorem 2.3 is that if ϕ_1, ϕ_2 are two L_p-semi-norms on $\mathbf{R}^n$ then so is the function $\phi : \mathbf{R}^n \to \mathbf{R}$ given by $\phi(\xi) = (\phi_1^p(\xi) + \phi_2^p(\xi))^{1/p}$. This result is well known in the case that ϕ_1 and ϕ_2 are "supported" by complemented subspaces and could also be derived for non-even integers from the results of Lévy and Krivine.

3. For every subspace V of $\mathbf{R}^n$ let $H_p^n(V)$ be the closed linear subspace of H_p^n generated by the functions $|\langle \xi, \cdot \rangle|^p$ with ξ in V. For every $1 \leqq i \leqq n$, let $C_p^n(i)$ be the cone generated by the positive functions in $H_p^n(V)$ with $\dim V \leqq i$.

LEMMA 3.1. *Let σ be a positive linear functional on H_p^n. Then the following conditions are equivalent*:

(i) *ϕ_σ is an L_p-norm on* $\mathbf{R}^n$.

(ii) *σ is strictly positive on* $C_p^n(1)$.

(iii) *All positive Borel measures on S^{n-1} representing σ are not supported by a proper subspace of* $\mathbf{R}^n$.

(iv) *There is a positive Borel measure on S^{n-1} that represents σ and is not supported by a proper subspace of* $\mathbf{R}^n$.

PROOF. (ii)$\to$(i): For every $\xi \in \mathbf{R}^n$ with $\xi \neq 0$, $|\langle \xi, \cdot \rangle|^p \in C_p^n(1) \setminus \{0\}$ and thus by (ii) $\phi_\sigma(\xi) = (\sigma(|\langle \xi, \cdot \rangle|^p))^{1/p} \neq 0$, which proves that ϕ_σ is a norm.

(i)$\to$(iii): Assume that a positive Borel measure μ on S^{n-1} represents σ and is supported by a proper subspace V of $\mathbf{R}^n$. Let $\xi \in \mathbf{R}^n \setminus \{0\}$ with $V \subset \xi^\perp$. Then

$$
\begin{aligned}
\phi_\sigma(\xi) &= (\sigma(|\langle \xi, \cdot \rangle|^p))^{1/p} \\
&= \left(\int_{S^{n-1}} |\langle \xi, x \rangle|^p d\mu(x) \right)^{1/p} \\
&= \left(\int_{S^{n-1} \cap \xi^\perp} |\langle \xi, x \rangle|^p d\mu(x) \right)^{1/p} \\
&= 0.
\end{aligned}
$$

(iii)$\to$(iv) is obvious.

(iv)$\to$(ii): Let μ be a positive Borel measure that represents σ and that is not supported by a proper subspace. Thus for every $\xi \in \mathbf{R}^n \setminus \{0\}$, $|\langle \xi, x \rangle|^p$ is not equal to 0 μ-a.e. and thus $\sigma(|\langle \xi, \cdot \rangle|^p) > 0$ for every $\xi \in \mathbf{R}^n \setminus \{0\}$. Since the functions $|\langle \xi, \cdot \rangle|^p$ generate the cone $C_p^n(1)$, the implication follows.

THEOREM 3.2. *Let $0 < p < \infty$ with $p \neq 2$. Then for every $1 < k \leqq n$, $C_p^n(k-1) \subsetneqq C_p^n(k)$.*

PROOF. Observe that it is sufficient to prove that $C_p^n(n-1) \subsetneqq C_p^n(n)$. To prove this strict inclusion we first assume that $p = 2l$ with l an integer larger than 1, and that $n \geqq 3$. Let $h : \mathbf{R}^n \to \mathbf{R}$ be given by

$$
h(x) = \sum_{i=1}^n \varepsilon(i) x_i^l \qquad \text{where } \varepsilon(i) = \begin{cases} +1 & i \text{ odd}, \\ -1 & i \text{ even}. \end{cases}
$$

Then $h^2(x)$ is a nonnegative homogeneous polynomial of degree $p = 2l$ and thus is in $C_p^n(n)$. We will prove that $h^2 \notin C_p^n(n-1)$. For that it is sufficient to prove that (using the finite dimensionality of H_p^n where p is an even integer) if $g \in H_p^n(V)$ for some $(n-1)$-dimensional subspace V of $\mathbf{R}^n$ and $0 \leqq g \leqq h^2$ then $g \equiv 0$. Let $\xi \in \mathbf{R}^n$ with $\xi^\perp = V$. As $g \in H_p^n(V)$, it has a representation $g(x) = \sum_{i=1}^t \alpha_i \langle \eta_i, x \rangle^{2l}$ with $\eta_i \in V$. In particular $g(x + \alpha\xi) = g(x)$, for every $\alpha \in \mathbf{R}$. As $g(x)$ is a polynomial in $x_1, \ldots, x_n$, in order to prove that $g \equiv 0$ it is sufficient to prove that g vanishes on some open subset of $\mathbf{R}^n$. We separate the proof into two parts: $h(\xi) \neq 0$ and $h(\xi) = 0$. If $h(\xi) \neq 0$ let $x \in \mathbf{R}^n$ be such that $h(\xi)h(x) < 0$. Obviously the set of these x's is open and nonempty. For such an x the function $h(x + \alpha\xi)$ which is continuous in α changes signs as α varies from 0 to $+\infty$. Therefore there is α for which $h(x + \alpha\xi) = 0$. Thus $0 \leqq g(x + \alpha\xi) \leqq h^2(x + \alpha\xi) = 0$ and therefore $g(x) = g(x + \alpha\xi) = 0$ which proves that g vanishes on an open nonempty set and thus $g \equiv 0$. For the case $h(\xi) = 0$ we first assume that l is an even integer. The set $\{x \in \mathbf{R}^n : \sum_{i=1}^n \varepsilon(i)\xi_i^{l-1} x_i \neq 0\}$ is nonempty and open. For each such x the function $h(x + \alpha\xi) = \alpha^{l-1} l \sum_{i=1}^n \varepsilon(i)\xi_i^{l-1} x_i +$ terms of lower order in α, changes sign when α varies from $-\infty$ to $+\infty$ and thus there is $\alpha \in \mathbf{R}$ for which $h(x+\alpha\xi)=0$ and as before this implies that $g(x)\equiv 0$. If l is an odd integer observe that the set $\{x : h(x)(\sum \xi_i^{l-1}\varepsilon(i)x_i)<0\}$ is obviously open and also nonempty (whenever $n \geqq 3$ and $l>1$). For such x, $h(x+\alpha\xi)$ changes sign as α goes from 0 to $+\infty$ and as before this implies that $g(x)=0$ and thus $g=0$.

Next assume that $p=2l$ is an even integer, $l \neq 1$ and $n=2$. Let $h(x)=(x_1-x_2)(x_1+x_2)^{l-1}$. Then $h^2 \in C_{2l}^2(2)$, and $h(1,1)=h(1,-1)=0$. If $g \in H_{2l}^2(V)$ for some one-dimensional subspace V, then $g(x)=a\langle\eta,x\rangle^2$. Therefore, if $0 \leqq g \leqq h^2$, it follows that $g \leqq 0$. This completes the proof for an even integer p.

Next assume that p is not an even integer. In order to prove that $C_p^n(n-1)$ is strictly contained in $C_p^n(n)$ it is enough to exhibit the existence of a linear functional σ on H_p^n that is positive on $C_p^n(n-1)$ but not on $C_p^n(n)$, i.e., to show the existence of a signed even measure μ on S^{n-1} that is not a positive measure such that for every $(n-1)$-dimensional subspace V of $\mathbf{R}^n$ and $f \in H_p^n(V)$, if $f \geqq 0$ then $\int_{S^{n-1}} f(x)d\mu(x) \geqq 0$. Let μ_1 be the orthogonal invariant measure on S^{n-1} and fix x_0 in S^{n-1}. Let $\mu_1(\varepsilon)$ be the restriction of μ to the set $\{x \in S^{n-1} : \min(\|x-x_0\|, \|x+x_0\|)<\varepsilon\}$, and let $\sigma=\sigma_\varepsilon$ be the linear functional on H_p^n associated with the signed measure $\mu=\mu_1-2\mu_1(\varepsilon)$. Let $0 \leqq f \in H_p^n(V)$ where V is an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. Then

$$\sigma(f)=\int_{S^{n-1}} f(y)d\mu(y)=\int_{S^{n-1}} f(P_V y)d\mu(y)$$

where P_V is the orthogonal projection on V. Letting μ_V be the orthogonal projection of the measure μ on V, i.e., for every Borel subset B of V, $\mu_V(B)=\mu(P_V^{-1}B)$ we have

$$\sigma(f)=\int_V f(y)d\mu_V(y);$$

it is now easy to verify using the homogeneity (of degree p) of functions in H_p^n that for sufficiently small ε, for every $(n-1)$-dimensional subspace V and every $0 \leqq f \in H_p^n(V)$, $\sigma(f) \geqq 0$. This completes the proof of Theorem 3.2.

REMARK 3.3. For $p=2$, $C_2^n(2)=C_2^n(n)$ since every positive quadratic form is the sum of squares of linear functionals.

COROLLARY 3.4. *For every $1 \leqq p<\infty$, $p \neq 2$, and every integer k, n with $2<k \leqq n$ there is a norm ϕ on $\mathbf{R}^n$ such that ϕ restricted to every $(k-1)$-dimensional subspace is an L_p-norm but there is a k-dimensional subspace for which ϕ is not an L_p-norm.*

PROOF. Let σ be a linear functional on H_p^n that is strictly positive on $C_p^n(k-1)$ but not positive on $C_p^n(k)$. The existence of such a linear functional follows from the strict inclusion in Theorem 3.2. Let $\phi_\sigma : \mathbf{R}^n \to \mathbf{R}$ be given by $\phi_\sigma(\xi) = (\sigma(|\langle \xi, \cdot \rangle|^p))^{1/p}$. For every subspace V of $\mathbf{R}^n$ with dim $V < k$, σ induces a strictly positive linear functional on $H_p^n(V)$ and thus ϕ_σ is an L_p-norm on V. As σ is not positive on $C_p^n(k)$ there is a subspace V of $\mathbf{R}^n$ of dimension k for which σ is not positive on $H_p^n(V)$ and thus is not an L_p-norm on V.

REMARK 3.5. The preceding corollary implies that there is no finite list of inequalities that characterizes embeddability in L_p, $p \neq 2$, since any finite list quantifies at most a fixed finite number of variables $x_1, \dots, x_k$ of the space X, and thus has to be satisfied by any normed space all of whose k-dimensional subspaces are L_p-spaces.

REMARK 3.6. The special case of Corollary 3.4 with $p = 4$, $k = n = 3$ has been demonstrated with an explicit example in theorem 8 of [6].

Added in proof. Recently, E. Gine and J. Zinn have independently proved (by different methods) Theorem 2.1 in the infinite dimensional set up, i.e., that if B is a separable Banach space then for every $p \in (0,\infty)$ that is not an even integer, the linear span of the set of functions on B, $\{|f(x)|^p : f \in B^*\}$ is dense in the space of symmetric, p-homogeneous, continuous functions on B for the topology of uniform convergence on compact sets.

The following argument shows how the infinite-dimensional result of Gine and Zinn follows from the finite-dimensional one. Assume that B is a separable Banach space. Let $C^p(B)$ be the closed subspace of $C(B)$ (the space of continuous functions on B, topology of uniform convergence on (norm) compact sets) spanned by the functions of the form $|f(x)|^p$ where $f \in B^*$. Then, for any $F \in C^p(\mathbf{R}^n)$ and $f_1, \dots, f_n \in B^*$, $F \circ (f_1, \dots, f_n) \in C^p(B)$ (the map $(f_1, \dots, f_n) : B \to \mathbf{R}^n$ is linear and continuous). Therefore, the results for $C^p(\mathbf{R}^n)$ assert on the one hand that $\max_{i=1}^n |f_i(x)|^p \in C^p(B)$ for any $f_1, \dots, f_n$ in B^* and thus that

$$\|x\|^p = \lim_{n\to\infty} \left(\max_{i=1}^n |f_i(x)|^p \text{ where } (f_i)_{i=1}^\infty \text{ is a w}^*\text{-dense sequence in the unit ball of } B^* \right)$$

is in $C^p(B)$, and on the other hand that $C^p(B)$ is a lattice (it is the closure of a lattice). Obviously, $C^p(B)$ separates points when considered as elements of continuous functions on P, where P is the space obtained from $S = \{x \in B : \|x\| = 1\}$ by identifying each point x of S with its antipodal point $-x$.

Thus $C^p(B)$ is the space of all symmetric, p-homogeneous, continuous functions on B.

Note that the preceding argument shows also that if $h \in C(P)$ with $\|h\|_\infty =: \sup\{|h(x)| : x \in P\}$ then for any compact subset K of P and any $\varepsilon > 0$ there is a linear combination $\Sigma_{i=1}^n \alpha_i |f_i(x)|^p$ ($\alpha_i \in \mathbf{R}$ and $f_i \in B^*$) such that

$$\sup\left\{\sum_{i=1}^n \alpha_i |F_i(x)|^p : x \in P\right\} \leqq \|h\|_\infty$$

and

$$\max\left\{\sum_{i=1}^n \alpha_i |f_i(x)|^p - h(x)| : x \in K\right\} < \varepsilon.$$

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